On a Spectral Representation for Correlation Measures in Configuration Space Analysis

Yuri M. Berezansky

Inst. Math., NASU, 252601 Kiev, Ukraine

Yuri G. Kondratiev

Inst. f. Angew. Math., Univ. Bonn, 53115 Bonn, Germany;
BiBoS, Univ. Bielefeld, 33615 Bielefeld, Germany; and
Inst. Math., NASU, 252601 Kiev, Ukraine

Tobias Kuna

Inst. f. Angew. Math., Univ. Bonn, 53115 Bonn, Germany; and
BiBoS, Univ. Bielefeld, 33615 Bielefeld, Germany

Eugene Lytvynov

Inst. f. Angew. Math., Univ. Bonn, 53115 Bonn, Germany; and
BiBoS, Univ. Bielefeld, 33615 Bielefeld, Germany

Abstract

The paper is devoted to the study of configuration space analysis by using the projective spectral theorem. For a manifold X, let Γ_X, resp. $\Gamma_{X,0}$ denote the space of all, resp. finite configurations in X. The so-called K-transform, introduced by A. Lenard, maps functions on $\Gamma_{X,0}$ into functions on Γ_X and its adjoint K^* maps probability measures on Γ_X into σ-finite measures on $\Gamma_{X,0}$. For a probability measure μ on Γ_X, $\rho_\mu := K^*\mu$ is called the correlation measure of μ. We consider the inverse problem of existence of a probability measure μ whose correlation measure ρ_μ is equal to a given measure ρ. We introduce an operation of $\star$-convolution of two functions on $\Gamma_{X,0}$ and suppose that the measure ρ is $\star$-positive definite, which enables us to introduce the Hilbert space $\mathcal{H}_\rho$ of functions on $\Gamma_{X,0}$ with the scalar product $(G^{(1)}, G^{(2)})_{\mathcal{H}_\rho} = \int_{\Gamma_{X,0}} (G^{(1)} \star \overline{G}^{(2)})(\eta)\, \rho(d\eta)$. Under a condition on the growth of the measure ρ on the n-point configuration spaces, we construct the Fourier transform in generalized joint eigenvectors of some special family $A = (A_\varphi)_{\varphi\in\mathcal{D}}$, $\mathcal{D} := C_0^\infty(X)$, of commuting selfadjoint operators in $\mathcal{H}_\rho$. We show that this Fourier transform is a unitary between $\mathcal{H}_\rho$ and the L^2-space $L^2(\Gamma_X, d\mu)$, where μ is the spectral measure of A. Moreover, this unitary coincides with the K-transform, while the measure ρ is the correlation measure of μ.

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To the memory of Professor Yuri L. Daletsky

1 Introduction

The configuration space Γ_X over a (non-compact) Riemannian manifold X is defined as the set of all locally finite subsets (configurations) in X. Such spaces as well as probability measures on them appear naturally in several topics of mathematics and physics. Let us mention only the theory of point processes [9, 6], classical statistical mechanics [22, 8], and nonrelativistic quantum field theory, e.g., [20, 21] and references therein.

An important tool in the study of configuration space analysis is the so-called K-transform. Roughly speaking, this transform maps functions defined on the space $\Gamma_{X,0}$ of finite configurations in X into functions defined on the space Γ_X of all configurations. Interpreting the algebra of functions on Γ_X as observables of our system, we may consider functions on $\Gamma_{X,0}$ as quasi-observables, from which we are able to reconstruct observables by using the K-transform. This special kind of observables is known in physics and called additive type observables, see [5]. A. Lenard was the first to recognize the operator nature of the K-transform [14, 15, 16]. Recently, this theory was reanalized and further developed in [10, 11, 12, 13], where the reader can find also many applications of this transform.

The adjoint K^* of the K-transform, defined by the formula

$$\int_{\Gamma_X} (KG)(\gamma)\,\mu(d\gamma) = \int_{\Gamma_{X,0}} G(\eta)(K^*\mu)(d\eta),$$

maps probability measures on Γ_X into σ-finite measures on $\Gamma_{X,0}$, and $\rho_\mu := K^*\mu$ is called the correlation measure of μ.

In several applications, a σ-finite measure ρ on $\Gamma_{X,0}$ appears as a given object and the problem is to show that this ρ can be seen as a correlation measure for a probability measure on Γ_X. Different types of sufficient conditions were given for this to hold. A. Lenard [15, 16] used essentially a positivity condition for the correlation measure, which allowed him to construct a linear positive functional and apply a version of the Riesz–Krein extension theorem. His conditions were also necessary. O. Macchi [18] (see also [6]) needed an additional condition in order to get an explicit construction of the measure on Γ_X.

The present paper is also devoted to this problem. As a first step, we utilize the idea of [10, 13], introducing the so-called $\star$-convolution on a space of functions on $\Gamma_{X,0}$ and demanding that ρ be $\star$-positive definite, that is,

$$\int_{\Gamma_{X,0}} (G \star \overline{G})(\eta)\,\rho(d\eta) \geq 0. \qquad (1)$$

Unlike the approach of [10, 13], where the authors prove a Bochner type theorem, we use a spectral approach. The condition (1) enables us to introduce in Section 2 the $\star$-convolution Hilbert space $\mathcal{H}_\rho$ of functions on $\Gamma_{X,0}$ with the scalar product

$$(G^{(1)}, G^{(2)})_{\mathcal{H}_\rho} := \int_{\Gamma_{X,0}} (G^{(1)} \star \overline{G}^{(2)})(\eta)\, \rho(d\eta).$$

Next, we follow the general strategy of representation of positive definite kernels and functionals by using the projective spectral theorem, see [2, 3, 4, 17]. We consider in the space $\mathcal{H}_\rho$ a family $(A_\varphi)_{\varphi\in\mathcal{D}}$ of Hermitian operators defined by the formula

$$(A_\varphi G)(\eta) := (\varphi \star G)(\eta)$$

on an appropriate domain. Here, $\mathcal{D} := C_0^\infty(X)$ is the nuclear space of all C^∞ functions on X with compact support, and each $\varphi \in \mathcal{D}$ is identified with the function on $\Gamma_{X,0}$ given as follows: $\varphi(\eta) := \varphi(x)$ if $\eta = \{x\}$ and $\varphi(\eta) := 0$ if the number of points in $\eta \in \Gamma_{X,0}$ is not equal to one.

Under a rather weak condition on the measure ρ, we show that the operators A_φ are essentially selfadjoint in $\mathcal{H}_\rho$ and their closures $A_\varphi^\sim$ constitute a family of commuting selfadjoint operators in $\mathcal{H}_\rho$. Moreover, these operators are shown to satisfy the conditions of the projective spectral theorem, and the Fourier transform in generalized joint eigenvectors of the family $(A_\varphi^\sim)_{\varphi\in\mathcal{D}}$ gives a unitary isomorphism between $\mathcal{H}_\rho$ and an L^2-space $L^2(\mathcal{D}', d\mu)$, where $\mathcal{D}'$ is the dual of $\mathcal{D}$ and μ is the spectral measure of the family $(A_\varphi^\sim)_{\varphi\in\mathcal{D}}$. Under this isomorphism, each operator $A_\varphi^\sim$ goes over into the operator of multiplication by the monomial $\langle\cdot, \varphi\rangle$. The corresponding Parseval inequality gives the required spectral representation of the functional determined by the measure ρ.

In Section 3, following an idea in [10], we prove, under an additional, natural condition on ρ, that the measure μ is concentrated actually on Γ_X. Notice that the configuration space can be considered as a subset of $\mathcal{D}'$ by identifying any configuration from Γ_X with a sum of delta functions having support in the points of the configuration. Moreover, the unitary constructed in Section 2 coincides now with the K-transform, since $\rho = \rho_\mu$ is the correlation measure of μ.

Finally, let us stress that the spectral approach not only gives an alternative way to find sufficient conditions for a measure to be a correlation measure, but also gives a new understanding of the K-transform as a *unitary* operator between the Hilbert spaces $\mathcal{H}_\rho$ and $L^2(\Gamma_X, d\mu)$ which has the form of Fourier transform.

2 The $\star$-convolution Hilbert space and the corresponding Fourier transform

Let X be a connected, oriented C^∞ (non-compact) Riemannian manifold. We denote by $\mathcal{D}$ the space $C_0^\infty(X)$ of all real-valued infinite differentiable functions on X with compact support. This space can be naturally endowed with a topology of a nuclear space, see e.g. [7].

Let $\mathcal{F}_0(\mathcal{D}) := \mathbb{C}$ and $\mathcal{F}_n(\mathcal{D}) := \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$, $n \in \mathbb{N}$, where $\mathcal{D}_{\mathbb{C}}$ denotes the complexification of the real space $\mathcal{D}$ and $\widehat{\otimes}$ stands for the symmetric tensor product. Notice that $\mathcal{F}_n(\mathcal{D})$ is the complexification of the space of all real-valued C^∞ symmetric functions on X^n with compact support. Next, we define

$$\mathcal{F}_{\mathrm{fin}}(\mathcal{D}) := \bigoplus_{n=0}^{\infty} \mathcal{F}_n(\mathcal{D})$$

to be the topological direct sum of the spaces $\mathcal{F}_n(\mathcal{D})$, i.e., an arbitrary element $G \in \mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ is of the form $G = (G^{(0)}, G^{(1)}, \dots, G^{(n)}, 0, 0, \dots)$, where $G^{(i)} \in \mathcal{F}_i(\mathcal{D})$, and the convergence in $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ means the uniform finiteness and the coordinate-wise convergence. In what follows, we will identify a $G^{(n)} \in \mathcal{F}_n(\mathcal{D})$ with the element $(0, \dots, 0, G^{(n)}, 0, 0, \dots) \in \mathcal{F}_{\mathrm{fin}}(\mathcal{D})$.

Next, we define the space $\ddot{\Gamma}_{X,0}$ of multiple finite configurations over X:

$$\ddot{\Gamma}_{X,0} := \bigsqcup_{n \in \mathbb{N}_0} \ddot{\Gamma}_X^{(n)}.$$

Here, $\mathbb{N}_0 := \{0, 1, 2, \dots\}$, $\ddot{\Gamma}_X^{(0)} := \{\varnothing\}$ and $\ddot{\Gamma}_X^{(n)}$, $n \in \mathbb{N}$, is the factor space

$$\ddot{\Gamma}_X^{(n)} := X^n / S_n$$

with S_n being the group of all permutations of $\{1, \dots, n\}$, which naturally acts on X^n:

$$\sigma(x_1, \dots, x_n) = (x_{\sigma(1)}, \dots, x_{\sigma(n)}), \qquad \sigma \in S_n. \tag{2}$$

Thus, an $\eta = [x_1, \dots, x_n] \in \ddot{\Gamma}_X^{(n)}$ is an equivalence class consisting of n elements each of which is a point in X (an n-point configuration in X with possibly multiple points).

Each $\ddot{\Gamma}_X^{(n)}$ is equipped with the factor topology generated by the topology on X^n, and $\ddot{\Gamma}_{X,0}$ is equipped then by the topology of disjoint union. It follows directly from the construction of $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ that each $G \in \mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ can be considered as the function on $\ddot{\Gamma}_{X,0}$ defined by

$$G(\varnothing) := G^{(0)},$$

$$G([x_1,\dots,x_n]) := G^{(n)}(x_1,\dots,x_n), \qquad n\in\mathbb{N}. \tag{3}$$

Notice that in the formula (3) we fixed, in fact, a numeration of the points in X defining the equivalence class, but the right hand side of (3) is independent of the numeration. Now, we will need a numeration once more to define the notion of summation over partitions of an equivalence class.

So, let $\eta=[x_1,\dots,x_n]$ be an equivalence class with a fixed numeration of points. To each nonempty subset ξ of the set $\{1,\dots,n\}$ there corresponds the equivalence class defined by the points x_i, $i\in\xi$. The $\xi=\varnothing$ as a subset of $\{1,\dots,n\}$ corresponds to the $\varnothing$ as an element of $\ddot\Gamma_X^{(0)}$. Thus, we will preserve the notation ξ for the corresponding element of $\ddot\Gamma_{X,0}$. For a function $F\colon(\ddot\Gamma_{X,0})^k\to\mathbb{C}$, we let

$$\sum_{(\xi_1,\dots,\xi_k)\in\mathcal{P}_k(\eta)} F(\xi_1,\dots,\xi_k) \tag{4}$$

denote the summation over all partitions $(\xi_1,\dots,\xi_k)$ of $\{1,\dots,n\}$. As easily seen, the result of the summation (4) is independent of the numeration.

Now, we define a convolution $\star$ as the mapping

$$\star\colon \mathcal{F}_{\mathrm{fin}}(\mathcal{D})\oplus\mathcal{F}_{\mathrm{fin}}(\mathcal{D})\to\mathcal{F}_{\mathrm{fin}}(\mathcal{D}) \tag{5}$$

given by

$$(G_1\star G_2)(\eta) := \sum_{(\xi_1,\xi_2,\xi_3)\in\mathcal{P}_3(\eta)} G_1(\xi_1\cup\xi_2)G_2(\xi_2\cup\xi_3), \tag{6}$$

where $\mathcal{P}_3(\eta)$ denotes the set of all partitions (ξ_1,ξ_2,ξ_3) of η in 3 parts.

Lemma 1 *$\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ with the operation $\star$ is a commutative nuclear algebra.*

Proof. For a class $\eta=[x_1,\dots,x_n]$, let $|\eta|:=n$ Since $G_1,G_2\in\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$, there exist $n_1,n_2\in\mathbb{N}_0$ such that $G_i(\eta)=0$ if $|\eta|>n_i$. Then, (6) implies that $(G_1\star G_2)(\eta)=0$ if $|\eta|>n_1+n_2$.

Next, we note that, for arbitrary $G_1^{(n_1+n_2)}\in\mathcal{D}^{\widehat{\otimes}(n_1+n_2)}$ and $G_2^{(n_2+n_3)}\in\mathcal{D}^{\widehat{\otimes}(n_2+n_3)}$, the function

$$G_1^{(n_1+n_2)}(x_1,\dots,x_{n_1},x_{n_1+1},\dots,x_{n_1+n_2})\times \\ \times G_2^{(n_2+n_3)}(x_{n_1+1},\dots,x_{n_1+n_2},x_{n_1+n_2+1},\dots,x_{n_1+n_2+n_3})$$

belongs to $\mathcal{D}^{\otimes(n_1+n_2+n_3)}$—the $(n_1+n_2+n_3)$-th tensor power of $\mathcal{D}$—and moreover, it depends continuously on G_1 and G_2. Hence, it is easy to see that $G_1\star G_2$ indeed belongs to $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ and that the operation (5) is continuous.

The commutativity of $\star$ follows directly from the definition. Thus, it remains only to show the associativity. It follows from (6) and an easy combinatoric consideration that

$$((G_1 \star G_2) \star G_3)(\eta) = \sum_{(\xi_1,\xi_2,\xi_3)\in\mathcal{P}_3(\eta)} (G_1 \star G_2)(\xi_1 \cup \xi_2) G_3(\xi_2 \cup \xi_3)$$

$$= \sum_{(\xi_1,\xi_2,\xi_3)\in\mathcal{P}_3(\eta)} \sum_{(\psi_1,\psi_2,\psi_3)\in\mathcal{P}_3(\xi_1\cup\xi_2)} G_1(\psi_1 \cup \psi_2) G_2(\psi_2 \cup \psi_3) G_3(\xi_2 \cup \xi_3)$$

$$= \sum_{(\xi_1,\xi_2,\xi_3)\in\mathcal{P}_3(\eta)} \sum_{(\psi_{11},\psi_{12},\psi_{13})\in\mathcal{P}_3(\xi_1)} \sum_{(\psi_{21},\psi_{22},\psi_{23})\in\mathcal{P}_3(\xi_2)} G_1(\psi_{11} \cup \psi_{12} \cup \psi_{21} \cup \psi_{22}) \times$$

$$\times G_2(\psi_{12} \cup \psi_{13} \cup \psi_{22} \cup \psi_{23}) G_3(\psi_{21} \cup \psi_{22} \cup \psi_{23} \cup \xi_3)$$

$$= \sum_{(\xi_1,\dots,\xi_7)\in\mathcal{P}_7(\eta)} G_1(\xi_1 \cup \xi_4 \cup \xi_6 \cup \xi_7) G_2(\xi_2 \cup \xi_4 \cup \xi_5 \cup \xi_7) G_3(\xi_3 \cup \xi_5 \cup \xi_6 \cup \xi_7).$$

Absolutely analogously, one arrives at the same result when calculating $(G_1 \star (G_2 \star G_3))(\eta)$. ■

We will need now also the (usual) space of finite configurations over X—denoted by $\Gamma_{X,0}$—which is defined as a subset of $\ddot{\Gamma}_{X,0}$ consisting of $\varnothing$ and all $\eta = [x_1, \dots, x_n] \in \ddot{\Gamma}_{X,0}$ such that $x_i \neq x_j$ if $i \neq j$. Each $\eta = [x_1, \dots, x_n] \in \Gamma_{X,0}$ can be identified with the set $\{x_1, \dots, x_n\}$. We have $\Gamma_{X,0} = \bigsqcup_{n\in\mathbb{N}_0} \Gamma_X^{(n)}$, where $\Gamma_X^{(n)}$ is the space of n-point configurations in X.

The space $\Gamma_{X,0}$ is endowed with the relative topology as a subset of $\ddot{\Gamma}_{X,0}$.

Let ρ be a measure on the Borel σ-algebra $\mathcal{B}(\Gamma_{X,0})$. Of course, ρ can be considered as a measure on $\mathcal{B}(\ddot{\Gamma}_{X,0})$ such that the (measurable) set $\ddot{\Gamma}_{X,0} \setminus \Gamma_{X,0}$ is of zero ρ measure. One sees that the restriction of ρ to $\ddot{\Gamma}_X^{(n)}$ is actually a measure on $\mathcal{B}_{\mathrm{sym}}(X^n)$. Here, $\mathcal{B}_{\mathrm{sym}}(X^n)$ denotes that sub-σ-algebra of the Borel σ-algebra $\mathcal{B}(X^n)$ consisting of symmetric sets, i.e., sets in X^n which are invariant with respect to the action (2) of the permutation group S_n on X^n. For example, for each Borel $\Lambda \in \mathcal{B}(X)$ we have $\Lambda^n \in \mathcal{B}_{\mathrm{sym}}(X^n)$.

We will suppose that ρ satisfies the following assumptions:

(A1) *Normalization*: $\rho(\Gamma_X^{(0)}) = 1$.

(A2) *Local finiteness*: For each $n \in \mathbb{N}$ and each compact subset $\Lambda \subset X$, we have

$$\rho(\Gamma_\Lambda^{(n)}) < \infty$$

(where $\Gamma_\Lambda^{(n)}$ denotes, of course, the n-point configuration space over Λ).

(A3) *Positive definiteness*: For each $G \in \mathcal{F}_{\text{fin}}(\mathcal{D})$

$$\int_{\Gamma_{X,0}} (G \star \overline{G})(\eta)\, \rho(d\eta) \geq 0,$$

where $\overline{G}$ is the complex conjugate of G.

Thus, it follows from (A2) and (A3) that

$$\mathcal{F}_{\text{fin}}(\mathcal{D}) \oplus \mathcal{F}_{\text{fin}}(\mathcal{D}) \ni (G_1, G_2) \mapsto a_\rho(G_1, G_2) := \int_{\Gamma_{X,0}} (G_1 \star G_2)(\eta)\, \rho(d\eta) \in \mathbb{C}$$

is a bilinear continuous form which is positive definite: $a_\rho(G, \overline{G}) \geq 0$. Therefore, by using the general technique, e.g., [2], Ch. 5, Sect. 5, subsec. 1, we can construct a nuclear factor-space

$$\widehat{\mathcal{F}}_{\text{fin}}(\mathcal{D}) := \mathcal{F}_{\text{fin}}(\mathcal{D}) / \{G' : a_\rho(G', \overline{G}') = 0\}, \tag{7}$$

consisting of factor classes

$$\widehat{G} = \{G' \in \mathcal{F}_{\text{fin}}(\mathcal{D}) : a_\rho(G - G', \overline{G} - \overline{G}') = 0\},$$

and then a Hilbert space $\mathcal{H}_\rho$ as the closure of $\widehat{\mathcal{F}}_{\text{fin}}(\mathcal{D})$ with respect to the norm generated by the scalar product $(\widehat{G}_1, \widehat{G}_2)_{\mathcal{H}_\rho} := a_\rho(G_1, \overline{G}_2)$. Thus, as a result we get a nuclear space $\widehat{\mathcal{F}}_{\text{fin}}(\mathcal{D})$ that is topologically, i.e., densely and continuously, embedded into the Hilbert space $\mathcal{H}_\rho$.

Now, for each $\varphi \in \mathcal{D}$, we define an operator $\mathcal{A}_\varphi$ acting on $\mathcal{F}_{\text{fin}}(\mathcal{D})$ as

$$\mathcal{A}_\varphi G := \varphi \star G, \qquad G \in \mathcal{F}_{\text{fin}}(\mathcal{D}),$$

and let A_φ be the operator in $\mathcal{H}_\rho$ with domain $\operatorname{Dom} A_\varphi = \widehat{\mathcal{F}}_{\text{fin}}(\mathcal{D})$ defined by

$$A_\varphi \widehat{G} := \widehat{\mathcal{A}_\varphi G} = \widehat{\varphi \star G}, \qquad G \in \mathcal{F}_{\text{fin}}(\mathcal{D}). \tag{8}$$

By Lemma 1,

$$\begin{aligned} a_\rho(\mathcal{A}_\varphi G_1, \overline{G}_2) &= \int_{\Gamma_{X,0}} ((\varphi \star G_1) \star \overline{G}_2)(\eta)\, \rho(d\eta) \\ &= \int_{\Gamma_{X,0}} (G_1 \star (\overline{\varphi \star G_2}))(\eta)\, \rho(d\eta) \\ &= a_\rho(G_1, \overline{\mathcal{A}_\varphi G_2}), \end{aligned}$$

and therefore the definition (8) makes sense due to [2], Ch. 5, Sect. 5, subsec. 2, which uses essentially the Cauchy–Schwartz inequality.

We strengthen now the condition (A2) by demanding the following:

(A2′) For every compact $\Lambda\subset X$, there exists a constant $C_\Lambda>0$ such that

$$\rho(\Gamma_\Lambda^{(n)})\le C_\Lambda^n \qquad \text{for all } n\in\mathbb{N}. \tag{9}$$

Lemma 2 *Let* (A1), (A2′), *and* (A3) *hold. Then the operators A_φ, $\varphi\in\mathcal{D}$, with domain $\widehat{\mathcal{F}}_{\mathrm{fin}}(\mathcal{D})$ are essentially selfadjoint in $\mathcal{H}_\rho$ and their closures, $A_\varphi^\sim$, constitute a family of commuting selfadjoint operators, where the commutation is understood in the sense of the resolutions of the identity.*

Proof. Let us show that, for any $G^{(n)}\in\mathcal{F}_n(\mathcal{D})$, $\widehat{G}^{(n)}$ is an analytical vector of each A_φ, i.e., the series

$$\sum_{k=0}^\infty \frac{\|A_\varphi^k\widehat{G}^{(n)}\|_{\mathcal{H}_\rho}}{k!}\,|z|^k,\qquad z\in\mathbb{C}, \tag{10}$$

has a positive radius of convergence. So, let us fix $\varphi\in\mathcal{D}$ and $G^{(n)}\in\mathcal{F}_n(\mathcal{D})$ and let Λ be a compact set in X such that $\operatorname{supp}\varphi\subset\Lambda$ and $\operatorname{supp}G^{(n)}\subset\Lambda^n$.

We will say that a measurable function G on $\ddot\Gamma_{X,0}$ has bounded support if there exists a compact set $\Lambda\subset X$ and $N\in\mathbb{N}$ such that $\operatorname{supp}G\subset\bigsqcup_{n=0}^N\ddot\Gamma_\Lambda^{(n)}$. The space of all bounded measurable functions with bounded support will be denoted by $B_{\mathrm{bs}}(\ddot\Gamma_{X,0})$. Evidently, the formula (6) can be extended to the case where $G_1,G_2\in B_{\mathrm{bs}}(\ddot\Gamma_{X,0})$.

Set now

$$\widetilde{\varphi}(\eta):=\sup_{x\in X}|\varphi(x)|\,\mathbf{1}_\Lambda(\eta),\qquad \widetilde{G}^{(n)}(\eta):=\sup_{\eta\in\Gamma_X^{(n)}}|G^{(n)}(\eta)|\,\mathbf{1}_{\Lambda^n}(\eta),$$

where $\mathbf{1}_Y(\cdot)$ denotes the characteristic function of a set Y. Denote by m the volume measure on X. Without loss of generality, we can suppose that $m(\Lambda)\ge1$. Let $\widetilde\rho_\Lambda$ be the measure on $\ddot\Gamma_{X,0}$ defined by

$$\widetilde\rho_\Lambda\restriction\ddot\Gamma_X^{(n)}:=C_\Lambda^n\,m^{\otimes n},$$

where C_Λ is the constant from (A2′) corresponding to the set Λ. Then, it is easy to see that

$$\begin{aligned}\|A_\varphi^k\widehat{G}^{(n)}\|_{\mathcal{H}_\rho}^2&=\int_{\Gamma_{X,0}}\big((\varphi^{\star k}\star G^{(n)})\star(\varphi^{\star k}\star\overline{G}^{(n)})\big)(\eta)\,\rho(d\eta)\\&\le\int_{\ddot\Gamma_{X,0}}\big((\widetilde\varphi^{\star k}\star\widetilde{G}^{(n)})^{\star2}\big)(\eta)\,\widetilde\rho_\Lambda(d\eta)\\&=\int_{\ddot\Gamma_{X,0}}\big((\widetilde{G}^{(n)})^{\star2}\star\widetilde\varphi^{\star2k}\big)(\eta)\,\widetilde\rho_\Lambda(d\eta).\end{aligned} \tag{11}$$

For any $R^{(n)}$ and f from $B_{\rm bs}(\ddot\Gamma_{X,0})$ which are only not equal to zero on $\ddot\Gamma_X^{(n)}$ and $\ddot\Gamma_X^{(1)}$, respectively, we have

$$(R^{(n)}\star f)([x_1,\dots,x_k]) = \begin{cases} \sum_{i=1}^{n+1} f(x_i)R^{(n)}([x_1,\dots,\hat x_i,\dots,x_{n+1}]), & \text{if } k=n+1,\\ \sum_{i=1}^{n} f(x_i)R^{(n)}([x_1,\dots,x_n]), & \text{if } k=n,\\ 0, & \text{otherwise.}\end{cases} \tag{12}$$

Here, $\hat x_i$ denotes the absence of x_i. Therefore, if additionally $R^{(n)}\ge 0$ and $f\ge 0$, then

$$\int_{\ddot\Gamma_{X,0}} (R^{(n)}\star f)(\eta)\,\widetilde\rho_\Lambda(d\eta) \le C_{\Lambda,f}(2n+1)\int_{\ddot\Gamma_{X,0}} R^{(n)}(\eta)\,\widetilde\rho_\Lambda(d\eta), \tag{13}$$

where

$$C_{\Lambda,f} := \max\Big\{\operatorname*{ess\,sup}_{x\in X} f(x), C_\Lambda\int_X f(x)\,m(dx)\Big\},$$

which yields

$$\int_{\ddot\Gamma_{X,0}} (R\star f)(\eta)\,\widetilde\rho_\Lambda(d\eta) \le 2C_{\Lambda,f}(n+1)\int_{\ddot\Gamma_{X,0}} R(\eta)\,\widetilde\rho_\Lambda(d\eta)$$

for each $R\in B_{\rm bs}(\ddot\Gamma_{X,0})$, $R\ge 0$, satisfying $R\restriction \ddot\Gamma_X^{(k)}=0$ if $k>n$.

Hence, by using (13), we get

$$\begin{aligned}
&\int_{\ddot\Gamma_{X,0}} \big((\widetilde G^{(n)})^{\star 2}\star\widetilde\varphi^{\star 2k}\big)(\eta)\,\widetilde\rho_\Lambda(d\eta)\\
&\quad\le 2C_{\Lambda,\widetilde\varphi}(2n+2k)\int_{\ddot\Gamma_{X,0}} \big((\widetilde G^{(n)})^{\star 2}\star\widetilde\varphi^{\star(2k-1)}\big)(\eta)\,\widetilde\rho_\Lambda(d\eta)\\
&\quad\le (2C_{\Lambda,\widetilde\varphi})^2(2n+2k)(2n+2k-1)\int_{\ddot\Gamma_{X,0}} \big((\widetilde G^{(n)})^{\star 2}\star\widetilde\varphi^{\star(2k-2)}(\eta)\big)\,\widetilde\rho_\Lambda(d\eta)\\
&\quad\le\dots\le (2C_{\Lambda,\widetilde\varphi})^{2k}\,\frac{(2n+2k)!}{(2n)!}\int_{\ddot\Gamma_{X,0}} (\widetilde G^{(n)})^{\star 2}(\eta)\,\widetilde\rho_\Lambda(d\eta).
\end{aligned} \tag{14}$$

Thus, (11) and (14) give

$$\|A_\varphi^k\widehat G^{(n)}\|_{\mathcal H_\rho} \le (2C_{\Lambda,\widetilde\varphi})^k\big((2n)!\big)^{-1/2}2^{n+k}(n+k)!\,\|\widetilde G^{(n)}\|_{\mathcal H_{\widetilde\rho_\Lambda}}.$$

Since

$$\sum_{k=0}^\infty \frac{(4C_{\Lambda,\widetilde\varphi})^k(n+k)!}{k!}\,|z|^k<\infty \qquad \text{if } |z|<(4C_{\Lambda,\widetilde\varphi})^{-1},$$

the analyticity of $\widehat{G}^{(n)}$ for A_φ is proven. By using Nelson's analytic vector criterium (e.g., [23], Sect. X.6, or [2], Ch. 5, Th. 1.7) we conclude that the operators A_φ are essentially selfadjoint on $\widehat{\mathcal{F}}_{\mathrm{fin}}(\mathcal{D})$.

Next, by Lemma 1 and (8), the operators A_{φ_1} and A_{φ_2}, $\varphi_1, \varphi_2 \in \mathcal{D}$, commute on $\widehat{\mathcal{F}}_{\mathrm{fin}}(\mathcal{D})$. Since the operator A_{φ_2} is essentially selfadjoint on $\widehat{\mathcal{F}}_{\mathrm{fin}}(\mathcal{D})$, the set

$$(A_{\varphi_2}^{\sim} - z\,\mathrm{id})\widehat{\mathcal{F}}_{\mathrm{fin}}(\mathcal{D}), \qquad z \in \mathbb{C},\ \Im z \neq 0,$$

is dense in $\mathcal{H}_\rho$. Next, again using Lemma 1 and (8), we get

$$(A_{\varphi_2}^{\sim} - z\,\mathrm{id})\widehat{\mathcal{F}}_{\mathrm{fin}}(\mathcal{D}) = ((\mathcal{A}_{\varphi_2} - z\,\mathrm{id})\mathcal{F}_{\mathrm{fin}}(\mathcal{D}))\widehat{\ } \subset \widehat{\mathcal{F}}_{\mathrm{fin}}(\mathcal{D}).$$

Therefore, the operators $A_{\varphi_1}^{\sim}$, $A_{\varphi_2}^{\sim}$, and

$$A_{\varphi_1}^{\sim} \restriction (A_{\varphi_2}^{\sim} - z\,\mathrm{id})\widehat{\mathcal{F}}_{\mathrm{fin}}(\mathcal{D})$$

have a total set of analytical vectors. Thus, by [2], Ch. 5, Th. 1.15, the operators commute in the sense of the resolutions of the identity. ■

Let $\mathcal{D}'$ denote the dual space of $\mathcal{D}$ and let $\mathcal{C}_\sigma(\mathcal{D}')$ be the cylinder σ-algebra on $\mathcal{D}'$ (see e.g. [2], Ch. 2, Sect. 1, subsec. 9).

Theorem 1 *Let a measure ρ on $(\Gamma_{X,0}, \mathcal{B}(\Gamma_{X,0}))$ satisfy the assumptions* (A1), (A2′), *and* (A3). *Then, there exists a probability measure μ on $(\mathcal{D}', \mathcal{C}_\sigma(\mathcal{D}'))$ and a unitary isomorphism*

$$K\colon \mathcal{H}_\rho \to L^2(\mathcal{D}', \mathcal{C}_\sigma(\mathcal{D}'), d\mu) := L^2(d\mu)$$

such that the image of each operator $A_\varphi^{\sim}$, $\varphi \in \mathcal{D}$, under K is the operator of multiplication by the monomial $\langle \varphi, \cdot \rangle$ in $L^2(d\mu)$:

$$K A_\varphi^{\sim} K^{-1} = \langle \varphi, \cdot \rangle \cdot\ . \tag{15}$$

The unitary K is defined first on the dense set $\widehat{\mathcal{F}}_{\mathrm{fin}}(\mathcal{D})$ in $\mathcal{H}_\rho$ by the formula

$$\widehat{\mathcal{F}}_{\mathrm{fin}}(\mathcal{D}) \ni \widehat{G} = (\widehat{G}^{(n)})_{n=0}^{\infty} \mapsto K\widehat{G} = (K\widehat{G})(\omega) = \sum_{n=0}^{\infty} \langle G^{(n)}, {:}\omega^{\otimes n}{:} \rangle \tag{16}$$

(*the series in* (16) *is actually finite*) *and then it is extended by continuity to the whole $\mathcal{H}_\rho$ space. Here, $G = (G^{(n)})_{n=0}^{\infty} \in \mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ is an arbitrary representative of $\widehat{G} \in \widehat{\mathcal{F}}_{\mathrm{fin}}(\mathcal{D})$, and for any $\omega \in \mathcal{D}'$, ${:}\omega^{\otimes n}{:} \in \mathcal{D}'^{\widehat{\otimes} n}$ is the n-th Wick power of ω defined by the recurrence relation*

$${:}\omega^{\otimes 0}{:} = 1, \quad {:}\omega^{\otimes 1}{:} = \omega,$$

$$\langle \varphi^{\otimes(n+1)}, {:}\omega^{\otimes(n+1)}{:} \rangle = \frac{1}{n+1}\big[\langle \varphi^{\otimes(n+1)}, {:}\omega^{\otimes n}{:}\widehat{\otimes}\omega \rangle - n\langle (\varphi^2)\widehat{\otimes}\varphi^{\otimes(n-1)}, {:}\omega^{\otimes n}{:} \rangle\big], \tag{17}$$

$$\varphi \in \mathcal{D}.$$

Remark 1 Let $\mathcal{F}^*_{\mathrm{fin}}(\mathcal{D})$ stand for the dual of $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$. This is the topological direct product of the dual spaces $\mathcal{F}_n(\mathcal{D}')=\mathcal{D}'^{\widehat{\otimes}n}_{\mathbb{C}}$ of $\mathcal{F}_n(\mathcal{D})$. Thus, an arbitrary element R of $\mathcal{F}^*_{\mathrm{fin}}(\mathcal{D})$ has the form $R=(R^{(n)})_{n=0}^\infty$ where $R^{(n)}\in\mathcal{F}_n(\mathcal{D}')$. Next, it follows from (7) that the dual $\widehat{\mathcal{F}}^*_{\mathrm{fin}}(\mathcal{D})$ of $\widehat{\mathcal{F}}_{\mathrm{fin}}(\mathcal{D})$ can be identified with the factor-space

$$\widehat{\mathcal{F}}^*_{\mathrm{fin}}(\mathcal{D})=\mathcal{F}^*_{\mathrm{fin}}(\mathcal{D})/\big\{\,R:\ll G,R\gg=0 \text{ for each } G\in\mathcal{F}_{\mathrm{fin}}(\mathcal{D}) \text{ such that } a_\rho(G,\overline{G})=0\,\big\}.$$

Here, $\ll\cdot,\cdot\gg$ denotes the dual pairing between the spaces $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ and $\mathcal{F}^*_{\mathrm{fin}}(\mathcal{D})$ (as well as the pairing between the spaces $\widehat{\mathcal{F}}_{\mathrm{fin}}(\mathcal{D})$ and $\widehat{\mathcal{F}}^*_{\mathrm{fin}}(\mathcal{D})$ below). Thus, each element $R\in\mathcal{F}^*_{\mathrm{fin}}(\mathcal{D})$ is a representative of some element $\widehat{R}\in\widehat{\mathcal{F}}^*_{\mathrm{fin}}(\mathcal{D})$. Define now

$$R(\omega):=(:\omega^{\otimes n}:)_{n=0}^\infty\in\mathcal{F}^*_{\mathrm{fin}}(\mathcal{D}),$$

and let $\widehat{R}(\omega)$ be the corresponding element of $\widehat{\mathcal{F}}^*_{\mathrm{fin}}(\mathcal{D})$. Then, the formula (16) can be rewritten in the form

$$\widehat{\mathcal{F}}_{\mathrm{fin}}(\mathcal{D})\ni\widehat{G}\mapsto K\widehat{G}=(K\widehat{G})(\omega)=\ll\widehat{G},\widehat{R}(\omega)\gg, \tag{18}$$

and hence (15) yields

$$\begin{aligned}\ll A_\varphi\widehat{G},\widehat{R}(\omega)\gg&=(K(A_\varphi\widehat{G}))(\omega)=\langle\varphi,\omega\rangle(K\widehat{G})(\omega)\\&=\langle\varphi,\omega\rangle\ll\widehat{G},\widehat{R}(\omega)\gg,\qquad \widehat{G}\in\widehat{\mathcal{F}}_{\mathrm{fin}}(\mathcal{D}).\end{aligned}$$

So, $\widehat{R}(\omega)$ is a generalized joint eigenvector of the family $A^\sim_\varphi$, $\varphi\in\mathcal{D}$, belonging to the joint eigenvalue $\omega\in\mathcal{D}'$, and the unitary K, written in the form (18), is the Fourier transform in generalized joint eigenvectors of this family (see [2], Ch. 3, for a detailed exposition of the general theory).

Proof of Theorem 1. We will use the standard technique of construction of the Fourier transform in generalized joint eigenvectors of a family of commuting self-adjoint operators [2, 17, 4]. In fact, the existence of a measure and a unitary K satisfying (15) and given by the formula (16) with some kernels $:\omega^{\otimes n}:\in\mathcal{D}'^{\widehat{\otimes}n}$ follows from the following lemma.

Lemma 3 1) *For each* $\varphi\in\mathcal{D}$, A_φ *is a linear continuous operator on* $\widehat{\mathcal{F}}_{\mathrm{fin}}(\mathcal{D})$.

2) *For an arbitrary fixed* $\widehat{G}\in\widehat{\mathcal{F}}_{\mathrm{fin}}(\mathcal{D})$, *the mapping*

$$\mathcal{D}\ni\varphi\mapsto A_\varphi\widehat{G}\in\widehat{\mathcal{F}}_{\mathrm{fin}}(\mathcal{D})$$

is linear and continuous.

3) *The vacuum* $\widehat{\Omega} = (1,0,0,\dots)\widehat{\ } \in \widehat{\mathcal{F}}_{\mathrm{fin}}(\mathcal{D})$ *is a strong cyclic vector of the family* $(A^{\sim}_{\varphi})_{\varphi\in\mathcal{D}}$, *i.e., the linear span of the set*

$$\{\widehat{\Omega}\} \cup \{A_{\varphi_1}\cdots A_{\varphi_n}\widehat{\Omega} \mid \varphi_i \in \mathcal{D},\, i=1,\dots,n,\, n\in\mathbb{N}\}$$

is dense in $\widehat{\mathcal{F}}_{\mathrm{fin}}(\mathcal{D})$.

Proof of Lemma 3. 1) and 2) Clear by Lemma 1.

3) Denote by $\Omega = (1,0,0,\dots)$ the vacuum in $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$. It suffices to show that the set

$$\{\Omega\} \cup \{\mathcal{A}_{\varphi_1}\cdots \mathcal{A}_{\varphi_n}\Omega \mid \varphi_i \in \mathcal{D},\, i=1,\dots,n,\, n\in\mathbb{N}\}$$

is dense in $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$.

Because of (12), we have on $\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$

$$\mathcal{A}_\varphi = \mathcal{A}^+_\varphi + \mathcal{A}^0_\varphi, \tag{19}$$

where $\mathcal{A}^+_\varphi$ is a creation operator:

$$\mathcal{A}^+_\varphi \psi^{\otimes n} = (n+1)\varphi\widehat{\otimes}\psi^{\otimes n}, \tag{20}$$

and $\mathcal{A}^0_\varphi$ is a neutral operator:

$$\mathcal{A}^0_\varphi \psi^{\otimes n} = n(\varphi\psi)\widehat{\otimes}\psi^{\otimes(n-1)}. \tag{21}$$

Therefore, taking to notice that the $\mathcal{A}^+_\varphi$'s are the usual creation operators, the cyclicity of Ω for the operators $\mathcal{A}_\varphi$ follows from the proof of Theorem 2.1 in [17], p. 65. ■

To finish the proof of the theorem, we need only to show that (17) holds. To this end, denote for $G\in\mathcal{F}_{\mathrm{fin}}(\mathcal{D})$ $KG := K\widehat{G}$. Then, upon (15), (16), (19)–(21),

$$\langle\varphi,\cdot\rangle K(\varphi^{\otimes n}) = K\mathcal{A}_\varphi\varphi^{\otimes n} = (n+1)K(\varphi^{\otimes(n+1)}) + nK((\varphi^2)\widehat{\otimes}\varphi^{\otimes(n-1)}),$$

which implies (17). ■

Corollary 1 *Under the conditions of Theorem* 1, *we have for each* $G\in\mathcal{H}_\rho$

$$\int_{\Gamma_{X,0}} G(\eta)\,\rho(d\eta) = \int_{\mathcal{D}'} KG(\omega)\,\mu(d\omega).$$

Proof. Since K is unitary, we have, for arbitrary $G_1, G_2\in\mathcal{H}_\rho$,

$$\int_{\Gamma_{X,0}} (G_1\star\overline{G}_2)(\eta)\,\rho(d\eta) = \int_{\mathcal{D}'}(KG_1)(\omega)(\overline{KG_2})(\omega)\,\mu(d\omega).$$

By setting in this formula $G_1 = G$ and $G_2 = \widehat{\Omega}$ and noting that, from one hand side, the vacuum is the identity element for the $\star$-convolution and on the other hand $K\widehat{\Omega}\equiv 1$, we get the corollary. ■

Remark 2 Let us consider the functional

$$L(\varphi;\omega) := e^{\langle \log(1+\varphi),\omega\rangle},$$

which is evidently analytical in φ in a neighborhood of zero in $\mathcal{D}_{\mathbb{C}}$ for each fixed $\omega \in \mathcal{D}'$. Then, by differentiating this functional and by using the recurrence relation (17), one can show that L is the generating functional of the Wick monomials $\langle \varphi^{\otimes n}, :\omega^{\otimes n}:\rangle$, i.e.,

$$L(\varphi,\omega) = \sum_{n=0}^{\infty} \langle \varphi^{\otimes n}, :\omega^{\otimes n}:\rangle$$

for φ from a neighborhood of zero (more exactly, for $\varphi \in \mathcal{D}_{\mathbb{C}}$ such that $\sup_{x\in X} |\varphi(x)| < 1$). Notice that the functional L is just the character in the generalized translation operator approach to Poisson analysis [1].

3 The measure ρ as a correlation measure

The configuration space Γ_X over X is defined as the set of all locally finite subsets (configurations) in X:

$$\Gamma_X := \{\gamma \subset X \mid |\gamma \cap \Lambda| < \infty \text{ for each compact } \Lambda \subset X\}.$$

Here $|A|$ denotes the cardinality of a set A. One can identify any $\gamma \in \Gamma_X$ with the positive Radon measure

$$\sum_{x\in\gamma} \delta_x \in \mathcal{M}(X),$$

where δ_x is the Dirac measure with mass in x, $\sum_{x\in\varnothing} \delta_x$:=zero measure, and $\mathcal{M}(X)$ stands for the set of all positive Radon measures on $\mathcal{B}(X)$. The space Γ_X can be endowed with the relative topology as a subset of the space $\mathcal{M}(X)$ with the vague topology, i.e., the weakest topology on Γ_X such that all maps

$$\Gamma_X \ni \gamma \mapsto \langle f,\gamma\rangle := \int_X f(x)\,\gamma(dx) = \sum_{x\in\gamma} f(x)$$

are continuous. Here, $f \in C_0(X)$ (:=the set of all continuous functions in X with compact support). We will denote by $\mathcal{B}(\Gamma_X)$ the Borel σ-algebra on Γ_X. In fact, Γ_X is a measurable subset of $\mathcal{D}'$ and the trace σ-algebra of $\mathcal{C}_\sigma(\mathcal{D}')$ on Γ_X (i.e., the σ-algebra on Γ_X consisting of intersections of sets from $\mathcal{C}_\sigma(\mathcal{D}')$ with Γ_X) coincides with $\mathcal{B}(\Gamma_X)$.

The following lemma gives a direct representation of the Wick powers $:\omega^{\otimes n}:$ in the case where $\omega = \gamma$ is a configuration.

Lemma 4 *For each $\gamma \in \Gamma_X$, we have*

$$:\gamma^{\otimes n}: = \sum_{\eta \Subset \gamma,\, |\eta|=n} \widehat{\bigotimes_{x\in\eta}} \delta_x, \tag{22}$$

where the summation is extended over all n-point subconfigurations from γ.

Proof. For $n = 0$ and $n = 1$ the formula evidently holds, and let us suppose that it holds for all $m \le n$. Then, upon (17)

$$\begin{aligned}\langle \varphi^{\otimes(n+1)}, :\gamma^{\otimes(n+1)}:\rangle &= \frac{1}{n+1}\big[\langle\varphi^{\otimes n}, :\gamma^{\otimes n}:\rangle\langle\varphi,\gamma\rangle - n\langle(\varphi^2)\widehat{\otimes}\varphi^{\otimes(n-1)}, :\gamma^{\otimes n}:\rangle\big]\\ &= \frac{1}{n+1}\bigg(\sum_{\eta\Subset\gamma,\,|\eta|=n}\prod_{y\in\gamma}\varphi(y) - \sum_{\eta\Subset\gamma,\,|\eta|=n}\sum_{x\in\eta}\varphi^2(x)\prod_{y\in\eta\setminus\{x\}}\varphi(y)\bigg)\\ &= \frac{1}{n+1}\sum_{\eta\Subset\gamma,\,|\eta|=n}\prod_{x\in\gamma}\varphi(x)\sum_{y\in\gamma\setminus\eta}\varphi(y) = \sum_{\eta\Subset\gamma,\,|\eta|=n}\varphi(y). \quad \blacksquare\end{aligned}$$

As a direct consequence of Lemma 4 and Corollary 1, we get

Proposition 1 *Suppose that, under the assumptions of Theorem 1, the measure μ has the configuration space Γ_X as a set of full measure. Then, the operator K coincides with the K-transform between the spaces of functions of finite and infinite configurations, while the measure ρ is the correlation measure of μ* [14, 15, 16, 10].

To restrict the measure μ to Γ_X, we need an additional condition on ρ, which is also not very restrictive.

(A4) Every compact $\Lambda \subset X$ can be covered by a finite union of open sets $\Lambda_1, \dots, \Lambda_k$, $k \in \mathbb{N}$, which have compact closures and satisfy the estimate

$$\rho(\Gamma_{\Lambda_i}^{(n)}) \le (2+\varepsilon)^{-n} \quad \text{for all } i = 1, \dots, k \text{ and } n \in \mathbb{N},$$

where $\varepsilon = \varepsilon(\Lambda) > 0$.

Suppose, for example, that a measure ρ on $\Gamma_{X,0}$ has density $\widetilde{\rho}$ with respect to the Lebesgue–Poisson measure

$$\lambda := \sum_{n=0}^{\infty} \frac{1}{n!} m^{\otimes n},$$

and suppose that this density fulfills the estimate

$$\operatorname*{ess\,sup}_{\eta\subset\Gamma_X^{(n)}} \widetilde{\rho}(\eta) \le n!\, C^n, \qquad n \in \mathbb{N},$$

for some constant $C > 0$. Then ρ satisfies trivially (A2′) as well as (A4). (We note that this situation where the measure ρ has density with respect to the Lebesgue–Poisson measure is typical in applications.)

Theorem 2 *Let a measure ρ on $(\Gamma_{X,0}, \mathcal{B}(\Gamma_{X,0}))$ satisfy the assumptions* (A1), (A2′), (A3), (A4), *and let μ be the probability measure on $(\mathcal{D}', \mathcal{C}_\sigma(\mathcal{D}'))$ constructed in Theorem* 1. *Then, Γ_X is of full μ measure.*

Proof. The proof is a modification of part of the proof of Theorem 5.5 in [10].

For a function $\varphi \in \mathcal{D}_\mathbb{C}$, define a function $e(\varphi, \cdot)$ on $\Gamma_{X,0}$ as follows:

$$\Gamma_{X,0} \ni \eta \mapsto e(\varphi, \eta) := \prod_{x \in \eta} \varphi(x) \in \mathbb{C}.$$

It follows from Remark 2 that

$$e^{\langle \varphi, \omega \rangle} = \sum_{n=0}^{\infty} \langle (e^\varphi - 1)^{\otimes n}, :\omega^{\otimes n}: \rangle,$$

where φ belongs to a neighborhood of zero in $\mathcal{D}_\mathbb{C}$, more exactly, if $\sup_{x \in X} |\varphi(x)| < \delta$ for some $\delta > 0$. Therefore,

$$e^{\langle \varphi, \cdot \rangle} = K e(e^\varphi - 1, \cdot). \tag{23}$$

Fix a compact $\Lambda \subset X$. Let $\mathcal{C}_{\sigma,\Lambda}(\mathcal{D}')$ denote the sub-σ-algebra of $\mathcal{C}_\sigma(\mathcal{D}')$ generated by the functionals of the form

$$\mathcal{D}' \ni \omega \mapsto \langle \varphi, \omega \rangle \in \mathbb{C}, \qquad \varphi \in \mathcal{D}(\Lambda),$$

where $\mathcal{D}(\Lambda)$ denote the subspace of $\mathcal{D}$ consisting of those φ having support in Λ. Next, let μ_Λ stand for the restriction of the measure μ to the sub-σ-algebra $\mathcal{C}_{\sigma,\Lambda}(\mathcal{D}')$.

Let now $\varphi \in \mathcal{D}_\mathbb{C}(\Lambda)$. It follows from (23) that

$$e(e^\varphi - 1, \cdot) \star e(e^{\overline{\varphi}} - 1, \cdot) = e(e^{\varphi + \overline{\varphi}} - 1, \cdot).$$

Therefore, by using (A2′), we see that there exists $\delta_\Lambda > 0$ such that $e(e^\varphi - 1, \cdot) \in \mathcal{H}_\rho$ provided $\sup_{x \in X} |\varphi(x)| \le \delta_\Lambda$. Thus, by Corollary 1,

$$\int_{\mathcal{D}'} e^{\langle \varphi, \omega \rangle} \, \mu_\Lambda(d\omega) = \int_{\Gamma_{X,0}} e(e^\varphi - 1, \eta) \, \rho(d\eta), \qquad \varphi \in \mathcal{D}_\mathbb{C}(\Lambda), \ \sup_{x \in X} |\varphi(x)| \le \delta_\Lambda. \tag{24}$$

Thus, the formula (24) gives the analytic extension of the Fourier transform of the measure μ_Λ in a neighborhood of zero.

Let us introduce now a mapping $\mathcal{R}$ which transforms the set of measurable functions on Γ_Λ into itself as follows:

$$(\mathcal{R}F)(\eta) := \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} F(\xi), \qquad \eta \in \Gamma_\Lambda.$$

Let now Λ satisfy the condition

$$\rho(\Gamma_\Lambda^{(n)}) \le (2+\varepsilon)^n, \qquad \varepsilon > 0. \tag{25}$$

Define on $\mathcal{B}(\Gamma_\Lambda)$ the set function

$$\widetilde{\mu}_\Lambda(A) := \int_{\Gamma_\Lambda} (\mathcal{R}\mathbf{1}_A)(\eta)\,\rho(d\eta).$$

Since $\sum_{\xi\subset\eta} 1 = 2^n$ if $|\eta| = n$, we conclude that the bound (25) implies that $\widetilde{\mu}_\Lambda$ is a signed measure. Therefore, for $\varphi \in \mathcal{D}_{\mathbb{C}}(\Lambda)$, we have

$$\int_{\Gamma_\Lambda} e^{\langle\varphi,\eta\rangle}\,\widetilde{\mu}_\Lambda(d\eta) = \int_{\Gamma_\Lambda} (\mathcal{R}e^{\langle\varphi,\cdot\rangle})(\eta)\,\rho(d\eta). \tag{26}$$

Direct calculation shows that

$$(\mathcal{R}e^{\langle\varphi,\cdot\rangle})(\eta) = e(e^\varphi - 1, \eta),$$

and therefore, we have from (24) and (26)

$$\int_{\mathcal{D}'} e^{\langle\varphi,\omega\rangle}\,\mu_\Lambda(d\omega) = \int_{\Gamma_\Lambda} e^{\langle\varphi,\eta\rangle}\,\widetilde{\mu}_\Lambda(d\eta), \qquad \varphi \in \mathcal{D}_{\mathbb{C}}(\Lambda),\ \sup_{x\in X}|\varphi(x)| \le \delta_\Lambda.$$

Therefore, $\widetilde{\mu}_\Lambda$ is a probability measure on Γ_Λ, and moreover it coincides with the restriction of the measure μ_Λ to the set Γ_Λ considered as a subset of $\mathcal{D}'$.

Hence

$$\mu(\widetilde{\Gamma}_\Lambda) = 1, \tag{27}$$

where $\widetilde{\Gamma}_\Lambda$ denotes the set of all $\omega \in \mathcal{D}'$ whose restriction to the set Λ is a finite sum of delta functions concentrated in Λ and having disjoint support.

Now, let Λ be an arbitrary compactum in X and let $\Lambda_1, \dots, \Lambda_k$ be open subsets of X as in (A4) corresponding to Λ. Since

$$\widetilde{\Gamma}_{\bigcup_{i=1}^k \Lambda_i} = \bigcap_{i=1}^k \widetilde{\Gamma}_{\Lambda_i},$$

we conclude that (27) holds for *each* compact $\Lambda \subset X$. From here, we immediately conclude that $\mu(\Gamma_X) = 1$. ■

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